(42) 5.3 THE FUNDAMENTAL THEOREM OF CALCULUS (B)

Example 1	Find $\int_0^{\frac{\pi}{3}} \tan x \sec^5 x \, dx$

Solution

$$I = \int_0^{\frac{\pi}{3}} \tan x \sec^5 x \, dx = \int_0^{\frac{\pi}{3}} \sec^4 x \cdot \sec x \tan x \, dx$$

$t = \sec x \qquad dt = \sec x \tan x \, dx$

$x = 0 \quad \rightarrow \quad t = \sec 0 = 1$

$x = \frac{\pi}{3} \quad \rightarrow \quad t = \sec\frac{\pi}{3} = 2$

$$I = \int_1^2 t^4 \cdot dt = \frac{1}{5}\left[t^5\right]_1^2 = \frac{1}{5}\left(2^5 - 1^5\right) = \frac{1}{5}(32-1) = \frac{31}{5}$$

Example 2 4 May 19, 1992	Evaluate the following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x} dx$

Solution

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\sin x}{\cos^2 x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\right) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2 x + \tan x \sec x) dx = \left[\tan x + \sec x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \tan\frac{\pi}{3} + \sec\frac{\pi}{3} - \left(\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right)$$

$$= \sqrt{3} + 2 - \left(1 + \sqrt{2}\right) = 1 + \sqrt{3} - \sqrt{2}$$

Example 3 17 January 8 ,1997	Evaluate the following integral $\int_0^{\frac{\pi}{4}} \frac{(1+\tan x)^8 \sec x}{\cos x} dx$

Solution

$$I = \int_0^{\frac{\pi}{4}} \frac{(1+\tan x)^8 \sec x}{\cos x} dx = \int_0^{\frac{\pi}{4}} (1+\tan x)^8 \sec^2 x \, dx$$

$t = 1 + \tan x \qquad dt = \sec^2 x \; dx$

$x = 0 \quad \to \quad t = 1 + 0 = 1$

$x = \frac{\pi}{4} \quad \to \quad t = 1 + \tan\frac{\pi}{4} = 1 + 1 = 2$

$$I = \int_1^2 t^8 \; dt = \frac{1}{9}\left[t^9 \right]_1^2 = \frac{1}{9}\,(2^9 - 1) = \frac{511}{9}$$

Example 4 31 June 5, 2008	Evaluate the following integral $\int_0^{\frac{\pi}{2}} (1+\sin t)^3 \cos t \, dt$

Solution

$$I = \int_0^{\frac{\pi}{2}} (1+\sin t)^3 \cos t \, dt$$

$u = 1 + \sin t \qquad du = \cos t \; dt$

$t = 0 \quad \to \quad u = 1 + \sin 0 = 1 + 0 = 1$

$t = \frac{\pi}{2} \quad \to \quad u = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$

$$I = \int_1^2 u^3 \; du = \frac{1}{4}\left[u^4 \right]_1^2 = \frac{1}{4}(2^4 - 1) = \frac{1}{4}(16 - 1) = \frac{15}{4}$$

Example 5 35 August 15, 2009	Prove that : $\int_1^4 \cos^2 x \; dx = 3 - \int_1^4 \sin^2 x \; dx$

Solution

$$I = \int_1^4 \cos^2 x \; dx = \int_1^4 (1 - \sin^2 x) \, dx = \int_1^4 dx - \int_1^4 \sin^2 x \; dx = \left[x \right]_1^4 - \int_1^4 \sin^2 x \; dx$$

$$= 4 - 1 - \int_1^4 \sin^2 x \; dx = 3 - \int_1^4 \sin^2 x \; dx$$

Example 6 36 January 17, 2010	Evaluate the following integral $\int_0^{\frac{\pi}{2}} \cos(\sin x)\cos x \; dx$

Solution

$$I = \int_0^{\frac{\pi}{2}} \cos(\sin x)\cos x \; dx$$

$t = \sin x \qquad dt = \cos x \; dx$

$x = 0 \quad \rightarrow \quad t = \sin 0 = 0$

$x = \frac{\pi}{2} \quad \rightarrow \quad t = \sin\frac{\pi}{2} = 1$

$$I = \int_0^1 \cos t \; dt = \Big[\sin t\Big]_0^1 = (\sin 1 - \sin 0) = \sin 1$$

Example 7 24 August 3, 2002	Evaluate the following integral $\int_{\frac{\pi^3}{8}}^{\pi^3} \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}} dx$

Solution

$$I = \int_{\frac{\pi^3}{8}}^{\pi^3} \frac{\sin\sqrt[3]{x}}{\sqrt[3]{x^2}} dx = \int_{\frac{\pi^3}{8}}^{\pi^3} \sin x^{\frac{1}{3}} \cdot x^{\frac{-2}{3}} dx$$

$t = x^{\frac{1}{3}} \qquad dt = \frac{1}{3} x^{\frac{-2}{3}} \; dx \qquad ⇨⇨ \quad 3\,dt = x^{\frac{-2}{3}} \; dx$

$x = \frac{\pi^3}{8} \quad \rightarrow \quad t = \left(\frac{\pi^3}{8}\right)^{\frac{1}{3}} = \frac{\pi}{2}$

$x = \pi^3 \quad \rightarrow \quad t = (\pi^3)^{\frac{1}{3}} = \pi$

$$I = 3\int_{\frac{\pi}{2}}^{\pi} \sin t \cdot dt = -3\Big[\cos t\Big]_{\frac{\pi}{2}}^{\pi} = -3\left(\cos\pi - \cos\frac{\pi}{2}\right) = -3(-1 - 0) = 3$$

Example 8 28 January 13. 2007	Evaluate the following integral $\int_0^{\frac{\pi}{2}} \sin^6 x \cos x \, dx$

Solution

$$I = \int_0^{\frac{\pi}{2}} \sin^6 x \cos x \, dx$$

$t = \sin x \qquad dt = \cos x \; dx$

$x = 0 \quad \rightarrow \quad t = \sin 0 = 0$

$x = \frac{\pi}{2} \quad \rightarrow \quad t = \sin \frac{\pi}{2} = 1$

$$I = \int_0^1 t^6 \; dt = \frac{1}{7}\left[t^7\right]_0^1 = \frac{1}{7}(1-0) = \frac{1}{7}$$

Homework

1	Evaluate	$\int_0^{\frac{\pi}{2}} \sqrt{\sin x + 1} \cos x \; dx$	16 June 6 .1996
2	Evaluate	$\int_0^{\frac{\pi}{2}} \cos x \; \sin^5 x \, dx$	5 July 13, 1992
3	Evaluate	$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^2} dx$	3 December 30, 1991
4	Evaluate	$\int_0^{\frac{\pi}{2}} \sqrt{\cos x + 1} \sin x \; dx$	
5	Evaluate	$\int_0^{\frac{\pi}{2}} \tan^2 x \; dx$	19 July 29, 2000
6	Evaluate	$\int_0^{\frac{\pi}{4}} \sin 2x \; \cos^7 2x \; dx$	20 January 3 ,2001
7	Evaluate	$\int_0^{\frac{\pi}{2}} \sin^5 x \cos x \, dx$	21 May 27. 2001